Háskóli Íslands
Raunvísindadeild
Eðlisfræði

Haust 2014

Eðlisfræði þéttefnis I

Dæmablað 10

Skilafrestur 18. November 2014 kl. 15:00

1. **Fermi surfaces in two dimensions** (10)

   Consider a two-dimensional system of nearly free electrons (weak periodic potential) with a square unit cell (lattice constant a). Determine the Fermi surfaces for the cases of 1, 2, 3, and 5 electrons per unit cell. To this end, first project the free electron Fermi surface into the 1st Brillouin zone and then think about at what points gaps open. Plot the Fermi surfaces in the Brillouin zone. It might be useful to make separate plots for the different bands. (Note: Semi-quantitative plots are OK, i.e., you need to get the topology right, the exact positions are not so important.)

2. **Electron gas in two dimensions** (20)

   For a free and independent electron gas in two dimensions

   (a) What is the relation between n and k_{F} in two dimensions ?

   (b) Prove that in two dimensions the free electron density of levels $D(E)$ is a constant independent of E for $E > 0$, and 0 for $E < 0$. What is the constant ?

   (c) Show that because $D(E)$ constant, every term in the Sommerfeld expansion for n vanishes except the $T = 0$ term. Deduce that $\mu = E_{\mathrm{F}}$ at any temperature.

   (d) Deduce from

   $$n = \int_{-\infty}^{\infty} dE D(E) f(E)$$

   that when $D(E)$ is as in (c), then

   $$\mu + k_{\mathrm{B}} T \ln\left(1 + \exp(-\mu/k_{\mathrm{B}}T)\right) = E_{\mathrm{F}}$$

(e) Estimate from the above equation the amount by which μ differs from $E_{\rm F}$. Comment on the numerical significance of this "failure" of Sommerfeld expansion, and on the mathematical reason for the "failure".

3. **The Kronig-Penney model** (20)

Consider an electron in 1D in the presence of the periodic potential (Kronig-Penney model)

$$U(x) = \sum_{m=-\infty}^{\infty} U_0 \Theta(x - ma)\Theta(ma + b - x)$$

(a) Restrict your attention to a single unit cell, and write down the boundary conditions for the wave function as required by Bloch's theorem.

(b) Solve the Schrödinger equation by constructing ψ(x) from plane waves and imposing suitable boundary conditions at $x = 0, b, a$. The results is a relation between the Bloch index k and the energy.

(c) Take the limit $b \longrightarrow 0$, $U_0 \longrightarrow \infty$ with $U_0 b \longrightarrow W_0 \frac{\hbar^2 a^{-2}}{m}$. Show that the condition for the Bloch index simplifies to

$$\cos(ka) = \frac{W_0}{qa}\sin(qa) + cos(qa)$$

where q is related to the eigenenergy $\mathcal{E}$ via $q = (2m\mathcal{E}/\hbar^2)^{1/2}$.

(d) Produce plots of the lowest two energy bands $\mathcal{E}_{nk}(n = 0, 1)$ in the limit of part (c) with $a = 1$, $m = 1$, $\hbar = 1$, and $W_0 = 0.5$.